Accuplacer Advanced Algebra Refresher

Math & Physics Resource Center

This refresher is not intended to teach new concepts in depth. Rather, its purpose is to remind students of concepts once learned in a previous college or high school course.

1 Rational Expressions

Rational expression is simply a more compact way of saying an algebraic expression involving fractions. The same rules apply as manipulating expressions with only numbers. It's important to remember the following:

- When adding and subtracting fractions, remember to always find a common denominator first.
- When multiplying fractions, simply multiply the numerators and denominators together.
- When dividing one fraction by another, it is equivalent to multiply by the reciprocal ("the flip") of the divisor.
- If you have rational expressions within another rational expression, it is often easier to simplify the numerator and denominator separately first. Then, treat the operation as division.

2 Factoring and Solving Polynomial equations

The first step in solving most polynomial equations is to set one side of the equation equal to 0.

When factoring polynomials the first step is to remember to pull out the **greatest common factor** between all the terms. Once this is done, there are several techniques which are useful to remember in factoring/solving polynomial equations, depending upon the resulting form:

- If the polynomial has two terms:
  - Check if it is the difference of two squares
    $a^2 - b^2 = (a+b)(a-b)$
  - Check if it is the difference or sum of two cubes
    $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
- If the polynomial has three terms, try to use the guess and check method by looking at the factors of the leading coefficient and the constant term.
  - Recall the **rational roots theorem** which states that any rational (written as a fraction) zero must be of the form $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading coefficient.
- If the polynomial has four terms, the easiest method is to factor by **grouping**.

Once a polynomial equation is factored, we apply the **zero product property** and set each factor to zero individually to solve the equation.

When a quadratic polynomial does not factor, there are several other techniques which can be used including: **completing the square**, and **the quadratic formula**.

3 Systems of Linear Equations

There are 3 primary methods for solving systems of linear equations:

- **Substitution** Solve one of the equations and for a variable and plug it into the other equation.
- **Elimination** Multiply each equation by a number so that when you add the two equations, one of the variables will cancel the other out.
- **Graphing** Rewrite the equations into slope intercept form and plot them. The point where they intersect is the solution. This method is most useful in two variable systems.

With three or more variables, it is usually easiest to try and eliminate one variable at a time until you have a system of two equations and two variables with which you can use any of the above three methods.

4 Non-linear inequalities

When solving non-linear inequalities, the first important step is to set one side of the inequality to zero. From this point there are two methods to approach, depending upon the type of expression for the inequality:

- If the expression is a **polynomial**, factor it as much as possible.
- If the expression is **rational**, first rewrite everything as a single fraction, then treat the numerator and denominator as polynomials and factor both.

Once everything is factored, identify the **zeroes** of both the numerator and the denominator. These are the **critical numbers**. We want to plot these on a number line, and test points between each of the numbers to determine on which intervals the inequality is true.

Once we have the intervals, it is important to know when they are closed (include the endpoints) or open (exclude the endpoints).

- If the inequality sign is $>$ or $<$, this **excludes** equality and so we also exclude the endpoints of the intervals in the numerator of a rational expression, or in polynomial. This is usually denoted with a (or).
- If the inequality sign is $\geq$ or $\leq$, this **includes** equality and so we include the endpoints the endpoints of the intervals in the numerator of a rational expression, or in polynomial. This is usually denoted with a square bracket [or].
- Always exclude the endpoints given from the denominator of a rational expression. Those numbers are undefined in the expression. (We can never divide by 0!)

5 Composition of Functions

Sometimes, it is useful to "compose" functions. What this means is to plug one function into another. For two functions $f(x)$ and $g(x)$ we define composition as follows:

$$(f \circ g)(x) = f(g(x))$$

Which is read as f composed with g.

Here are some examples of two functions being composed with one another:

- $f(x) = \sqrt{x}$, $g(x) = x + 5$, $(f \circ g)(x) = \sqrt{x+5}$
- $g(x) = x + 7$, $f(x) = x^3 + 12$, $(g \circ f)(x) = x^3 + 19$

6 Inverse Functions

We say that two functions are inverses of one another if their composition undoes each other. What this means precisely is:

$$(f \circ g)(x) = x$$

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It is customary to write the inverse in terms of the other function as $f^{-1}(x)$. From above, $g(x) = f^{-1}(x)$. (Both of the above conditions must be met for g to be the inverse of f.)

Usually what we care about is the existence of an inverse function and being able to compute it (if it does indeed exist). The easiest method to check if an inverse exists is to use the **horizontal line test**. If the graph of a function does not cross any horizontal line (straight across) more than once, then the function has an inverse. Sometimes it is necessary to restrict the domain of a function in order for it to have an inverse.

To actually compute the inverse of a function the easiest method is to set the function value equal to y, and then exchange x and y in the equation. When we solve for y algebraically in the resulting equation we obtain $f^{-1}(x)$ directly. To check if you obtained the correct answer, simply compose f^{-1} with f and simplify as much as possible to see if you get x.

7 Logarithms and Exponentials

The logarithm has the following equivalence to an exponential expression:

$$a^x = b \iff \log_a(b) = x$$

This is read as: "Log base a of b is equal to x". By definition, the logarithm is the inverse of the exponential function with the same base. There are several shorthands commonly used for bases used frequently in mathematics and other areas:

- $\ln(x) = \log_e(x)$, where e is Euler's number ($e = 2.71828...$). This is commonly called the "Natural Logarithm".
- $\log(x) = \log_{10}(x)$

The logarithm also has several nice properties which we can use to manipulate them:

- $\log(ab) = \log(a) + \log(b)$
- $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$
- $\log(a^n) = n \log(a)$

Most calculators only have log base 10 and log base e functions. If you want to numerically approximate something in an arbitrary base, it is necessary to use one of these in conjunction with the change of base formula:

$$\log_a(b) = \frac{\ln(b)}{\ln(a)} = \frac{\log(b)}{\log(a)}$$

This formula applies to any arbitrary third base, it is simply most common to use one of those given above.

8 Matrices

A matrix is a collection of numbers ordered into rows and columns. An $n \times m$ matrix has n rows and m columns. An example of a 2×2 matrix is:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Matrix arithmetic has three operations:

- **Scaling** multiplying each entry by a single number.

- **Addition/Subtraction** Add/subtract the entries which are in the same location in each matrix. Matrices with different dimensions cannot be added or subtracted!
- **Matrix Multiplication** For a product of two matrices A and B, we multiply the rows of A by the columns of B. This means that for a product of two matrices to exist the number columns from the left must equal the number of rows on the right. Matrix multiplication is associative, but not commutative! $AB \neq BA$ necessarily.

A common problem for square matrices is to compute the determinant. The determinant for a 2×2 square matrix is given as:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

The determinant is only defined for square matrices! It also allows us to compute the inverse matrix for a 2×2.

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Note that if the determinant of a matrix is 0 then the inverse does not exist. The product of matrix and its inverse is always the the identity matrix I for the dimensions of that matrix. The 2×2 identity is:

$$A^{-1}A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The determinant of a 3×3 matrix is a little bit more complicated:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Visually, we pick an element and eliminate the row and column contained by it and take the determinant of the resulting matrix. You can see this if you lay a pencil and your finger on the row and column. Multiplying the entry by determinant of the new matrix. Take note also that the middle term is subtracted from the other two.

One of the major reasons for studying matrices is because of their equivalence to systems of linear equations. For example, this system:

$$\begin{cases} 4x + 5y = 6 \\ 3x - 2y = 7 \end{cases}$$

Could be rewritten in the matrix form:

$$\begin{pmatrix} 4 & 5 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$$

So solving the system amounts to computing the inverse of the matrix and multiplying to both sides.

9 Coordinate Plane and Distances

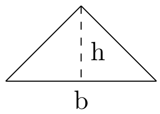
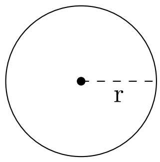
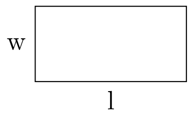
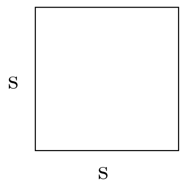
The distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

10 Common Geometric Figures

There are numerous common geometric figures that is useful to be acquainted with, a few are given below. Recall the following as well:

- **Area** is the size of the interior of a figure
- **Perimeter** is the length around the exterior of a figure

	Shape	Area	Perimeter
	Triangle	$\frac{1}{2}bh$	
	Circle	πr^2	$2\pi r$
	Rectangle	$l \times w$	$2l + 2w$
	Square	s^2	$4s$

11 Trigonometric Functions and their inverses

Trigonometric functions take as their inputs acute angles ($< 90^o$) of right triangles and give as outputs ratios of the sides of the triangle. Remember the acronym **SOH-CAH-TOA** to help remind you which sides go to which trigonometric function:

- **sin** or **Sine** is the opposite side over hypotenuse
- **cos** or **Cosine** is adjacent side over the hypotenuse
- **tan** or **Tangent** is the opposite over the adjacent side

Another useful fact is that:

$$\tan x = \frac{\sin x}{\cos x}$$

In addition we also have **reciprocal** trigonometric functions defined as follows:

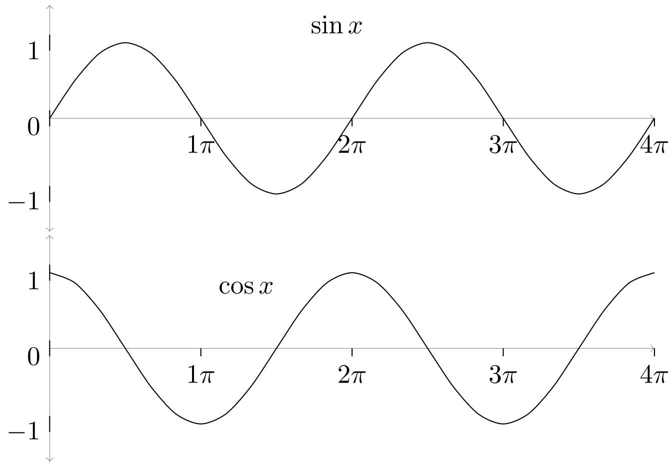
- **Secant** is the reciprocal of **Cosine**, $\sec x = \frac{1}{\cos x}$
- **Cosecant** is the reciprocal of **Sine**, $\csc x = \frac{1}{\sin x}$
- **Cotangent** is the reciprocal of **Tangent**, $\cot x = \frac{1}{\tan x}$

One important fact is that the graphs of Sine and Cosine are periodic waves, whose values are restricted between -1 and 1. This means that if when we define the inverse of either, we must restrict the domain of the original functions. The inverse trigonometric functions take a number between -1 and 1 as input and give as their output a corresponding angle. These restrictions are summarized in the table below:

Function	Domain	Range
$\sin^{-1}(x)$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

12 Trigonometric Graphs

The graphs of all trigonometric functions are periodic with periods of either π or 2π. The graph below shows basic **sine** and **cosine** curves. They both have period 2π, but note that **sine** begins at 0, while **cosine** begins at 1.



In general, we can also plot equations using graph transformations. When we do this, there are five important characteristics to keep in mind:

- **Period** how often does the graph repeat itself.
- **Phase Shift** does the graph have a horizontal shift to the left or the right.
- **Vertical Shift** does the graph move up or down from normally being centered at $y = 0$.
- **Amplitude** Normally sine and cosine oscillate between -1 and 1. The amplitude is the **positive** number that describes half the distance from a peak to a trough of a sine or cosine wave.
- **Reflection** is the graph reflected across its axis of symmetry?

In order to systematically analyze equations using graph transformations, we use the following form which tells us about the characteristics we care about:

$$f(x) = A\sin(Bx + C) + D$$

- **A** the absolute value of A will always be the amplitude of the graph. If A is negative then the graph is reflected across its axis of symmetry. (This axis of symmetry depends on the vertical shift of the graph.)
- **B** this number effects the period of the graph. The period is always $\frac{2\pi}{B}$
- **C** this number represents the phase shift, or how much the graph is move right or left from its original motion.
- **D** this number represents the vertical shift. The axis of symmetry for the graph is always $y = D$.

It's important to know the basic shapes of sine and cosine. It may also be helpful to know the shapes of tangent, cotangent, secant, and cosecant graphs.

13 Trigonometric Equations

When solving trigonometric equations, always keep in mind the **Pythagorean identity** and its related equations:

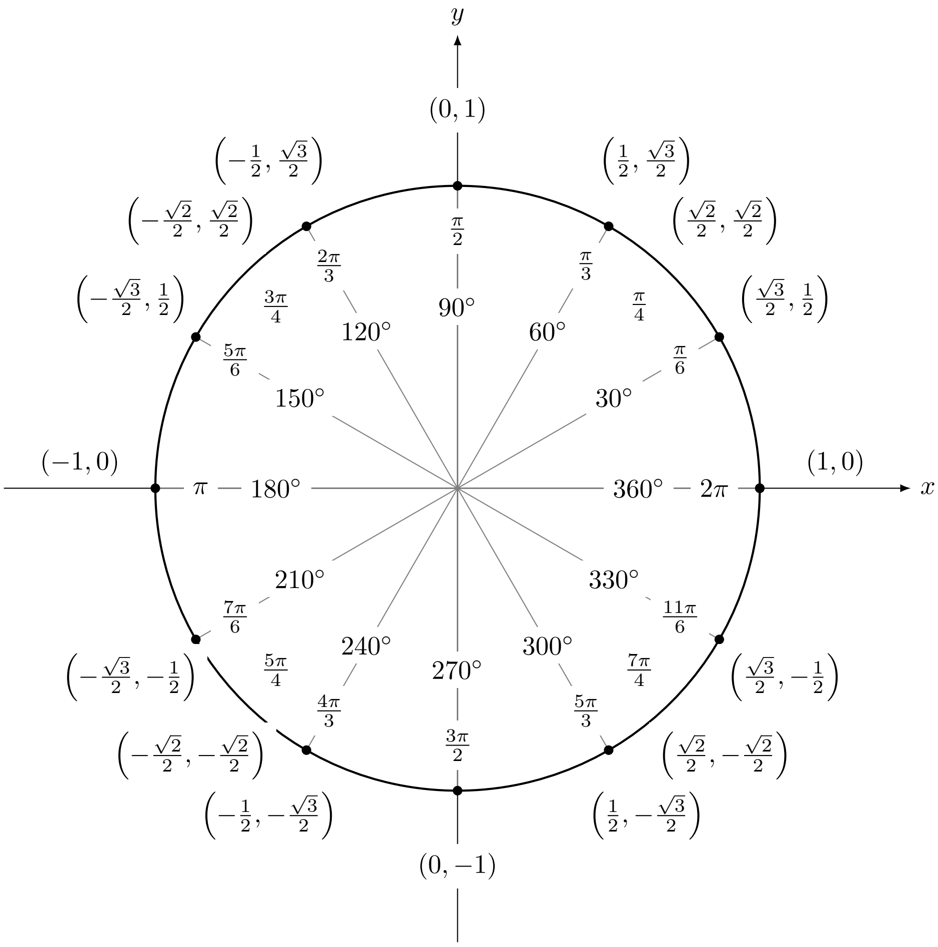
$$\sin^2(x) + \cos^2(x) = 1$$

$$\tan^2(x) + 1 = \sec^2(x)$$

$$\cot^2(x) + 1 = \csc^2(x)$$

The second two equations can be derived by dividing the Pythagorean identity by $\cos^2(x)$ or $\sin^2(x)$, respectively.

It is also useful to know the special angles for each triangle on the unit circle. In particular, memorizing the first quadrant angles and being able to apply them to the other three quadrants can help in solving many problems. A full unit circle is given below. Recall that cosine is the x-value of a particular coordinate, and sine is the y-value of a particular coordinate.



Solving most trigonometric equations is nearly equivalent to many factoring problems. Sometimes, you may obtain an equation of quadratic form in terms of a particular trigonometric function. In this case, it is sometimes easier to substitute the function in question with its own variable. Such as: $u = \sin(x)$. Once you make a substitution, factor it as much as possible before plugging your substitution back in.